A NEW METHOD FOR DETERMINING POSITION ERRORS OF PLANAR MECHANISMS INCLUDING DIMENSIONAL VARIATIONS IN ITS LINKAGES

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Abstract. The kinematic analysis of planar mechanisms is a very important tool to predict the position, velocity and acceleration of planar mechanisms. In fact, those results vary due to dimensional variations that the linkages undergo while they are manufactured. Dimensional variations arise due to tolerances that designers chose during the design phase of the mechanisms. If the projected tolerances are very precise, the production costs can rise to an undesirable magnitude. To reduce the production costs it is necessary to apply bigger tolerances which generate errors in the position of the mechanism projected. This paper presents a new method to predict the position errors of planar mechanisms including its dimensional variations. The variations in the position are included in the vectors that characterize the linkages and they are approximated by means of Taylor series. The equation that represents the position of the linkages is obtained by close loop vector, this equation corresponds to the residue of the close loop vector and it is nonlinear. A deterministic solution for nonlinear equation is obtained by maximizing errors with respect to each geometric parameter. A four bar mechanism is used as an example to explain in detail the new method, the results of determined position are compared with results obtained by the Direct Linearization Method (DLM) which shows that the proposed deterministic solution predicts bigger errors.

Keywords: Position error, dimensional variations, mechanisms, kinematical analysis.

1. INTRODUCTION

The industries are constantly improving the quality of the products and further they are reducing production costs to be more competitive in the market. The best option is to automate their processes to manufacture products continuously in large volumes. Automation requires a variety of mechanisms designed to accomplish different tasks with more efficiency than human beings. For the mechanisms to do their job successfully, they should be designed in a proper way to predict their behavior with dimensional requirements using kinematic analysis. But often the predictions of these tests are not accurate and it can vary in the reality. That is due to that the designer can assigns relatively small and precise tolerances to ensure that the product will function appropriately, however this creates higher production costs. If the designer to assign larger tolerances for each component, it could reduces costs, nevertheless it can generate a large loss in quality and product malfunction if the behavior is not predicted. The kinematic analysis of mechanisms is a mathematical tool very useful in predicting the movement of the elements that make up the mechanisms. The results obtained in the theoretical kinematic analysis can be inaccurate if they are compared with the kinematic results obtained experimentally. The errors obtained in the kinematic analysis are generated by the variables involved in the manufacture of items such as geometric tolerances and defects (Chase et al., 1995). These defects can accumulate and spread throughout the movement of the mechanism, causing significant changes in position of the elements when the mechanism is in operation (Dabling, 2001). The theoretical values taken into account in the kinematic analysis (lengths without tolerances and angular variations) do not include variations or geometric imperfections of the elements. Therefore, when performed kinematic analysis of a mechanism it is necessary to include and consider all the geometric variations of the elements of the mechanism to obtain results closer to the results obtained experimentally (Leishman and Chase, 2010).

In the past many researchers have worked with variation analysis in mechanisms, such as Lee and Gilmore (1991) which developed a method that estimates the means and variances of the velocity and acceleration for planar kinematic chains considering the effect of tolerances on link length. On the other hand, Huo and Chase (1996) demonstrated that the variations in the position present relationship with the velocity equation of a mechanism. And they also show the equivalency of the Jacobian matrix with the sensitivity matrix obtained for the position. Gao et al. (1998) established the Direct Linearization Method (DLM) to predict dimensional variations in the size of static assemblies, they estimate dimensional variations as geometric tolerances. The DLM presents a closed form solution in which the dimensional variations are obtained by an approximation of Taylor series on the equation obtained with closed loop vector. The main
The objective of this method is to predict small changes in the position, velocity and acceleration obtained with conventional kinematic analysis (see Leishman and Chase, 2010).

In this study was developed a method that permits to obtain position errors from a equation error obtained with closed kinematical chains. The dimensional variations are included in each vector that represents the linkages. The proposed method is based on the maximization of the error of each parameter of the mechanism and it is solved as a sum of maximum errors. So, the method maximizes the error of the position in the mechanism. The results show that the position errors obtained with the developed method are bigger than those obtained with DLM.

2. DIMENSIONAL VARIATIONS IN VECTORS

2.1. Approximation of Vectors with Dimensional Variations

The representation of vectors used frequently in vector analysis not considers small variations in its dimensional parameters, as it is shown in Fig. 1a. Figure 1b) shows a planar vector with dimensional and geometric changes, such as variation on the length (dr) and variation of the angular position (dθ). The magnitude of the vector of Fig. 1b) is defined as a function \( R(\theta + d\theta, r + dr) \), such that \( r, dr, \theta \) and \( d\theta \in \mathbb{R} \).

The vector \( R(\theta + d\theta, r + dr) \) shown in Fig. 1b) can be represented in polar form (see Norton, 2001) using the following expression

\[
R(\theta + d\theta, r + dr) = (r + dr)e^{j(\theta + d\theta)}.
\]

Equation (1) can be approximated by a Taylor series truncated in the second term about the point \((\theta, r)\) and evaluated in \( R(\theta, r) \) such that

\[
R(\theta + d\theta, r + dr) = R(\theta, r) + \frac{\partial R(\theta, r)}{\partial \theta} d\theta + \frac{\partial R(\theta, r)}{\partial r} dr.
\]

Simplifying the terms and carrying out partial derivatives \( \partial R(\theta, r) / \partial \theta \) and \( \partial R(\theta, r) / \partial r \) on Eq. (2), we obtain an expression that describes approximately the vector \( R(\theta + d\theta, r + dr) \) as

\[
R(\theta + d\theta, r + dr) \approx re^{j\theta} + rde^{j\theta} + dre^{j\theta}.
\]

The approximation of \( R(\theta + d\theta, r + dr) \) is used to represent vectors with variations and thereby it is possible to apply the vector algebra on vectors with variations. In Eq. (3) can be applied Euler's identity to represent \( e^{j\theta} \) in its real and imaginary components (see Norton, 2001).

3. KINEMATIC ANALYSIS OF PLANAR MECHANISMS INCLUDING DIMENSIONAL VARIATIONS

3.1. Theoretical Example: Four bar mechanism
The planar mechanism shown in Fig. 2a) is a four-bar mechanism; each link is connected by rotational planar joints. The link number one is considered fixed or ground. The orientations of each link are $\theta_1, \theta_2, \theta_3$ and $\theta_4$ and the dimensions of each link are denoted as $r_1, r_2, r_3$ and $r_4$. The subscripts 1, 2, 3 and 4 represent the number of link. $\theta_3$ and $\theta_4$ are unknown parameters for any configuration of the mechanism (for this case $\theta_2$ will be an orientation initially known).

Figure 2. a) Four bar mechanism. b) Close loop vector. c) Close loop vector with dimensional variations.

3.2. Position Equations without Dimensional Variations

To describe the position of the mechanism (see Fig. 2a) each link can be represented by a vector (see Fig. 2a), forming a closed loop vector with all the links. Figure 2b) shows the vector representation of the 4-link mechanism using a closed loop chains of vectors. The closed loop vector formed by vectors $\vec{r}_1(r_1, \theta_1), \vec{r}_2(r_2, \theta_2), \vec{r}_3(r_3, \theta_3)$ and $\vec{r}_4(r_4, \theta_4)$ represent the position of the links and it can be written as a vector sum such that

$$\vec{r}_1(r_1, \theta_1) + \vec{r}_2(r_2, \theta_2) + \vec{r}_3(r_3, \theta_3) - \vec{r}_4(r_4, \theta_4) = 0,$$

(4)

where $\vec{r}_i(r_i, \theta_i) = r_ie^{i\theta_i}, \forall i = 1, 2, 3, 4$(polar representation). Applying the polar representation to Eq. (4) we can obtain the solutions for the parameters $\theta_i$ and $\theta_4$ such that

$$\theta_i = \frac{\tan^{-1}\left( \frac{-2AB \pm \sqrt{(2AB)^2 - 4(B^2 - D^2)(A^2 - D^2)}}{2(B^2 - D^2)} \right)}{r_i},$$

$$\theta_4 = \tan^{-1}\left( \frac{r_1 \cos(\theta_1) + r_2 \cos(\theta_2) + r_3 \cos(\theta_3)}{r_4} \right),$$

(5)

where $A = r_1 \cos(\theta_1) + r_2 \cos(\theta_2), B = r_1 \sin(\theta_1) + r_2 \sin(\theta_2)$ and $D = (r_3^2 - r_4^2 - A^2 - B^2) / 2r_4$. Equations (5) resolve the angular position for each entry $\theta_i$.

3.3. Position Equations with Dimensional Variations

For the four bar mechanism shown in Fig. 2a) are included dimensional variations $dr_1, dr_2, dr_3, dr_4$ and angular variations $d\theta_1, d\theta_2, d\theta_3, d\theta_4$ on each vector that represents each link (Fig. 2c)). Variations $d\theta_1, d\theta_2, d\theta_3$ and $d\theta_4$ are variational parameters known and these are obtained by projections of manufacture (estimated measures) or geometric tolerances defined for the mechanism. $d\theta_1$ and $d\theta_4$ are unknown angular variations for any position of the mechanism. Determining the vector sum of the representation shown in Fig. 2c) we determine that

$$\vec{r}_1(r_1 + dr_1, \theta_1 + d\theta_1) + \vec{r}_2(r_2 + dr_2, \theta_2 + d\theta_2) + \vec{r}_3(r_3 + dr_3, \theta_3 + d\theta_3) - \vec{r}_4(r_4 + dr_4, \theta_4 + d\theta_4) = H,$$

(6)

where $H$ is a residue of the vector sum, which should be approximately zero in each position configuration of the mechanism. Each vector $\vec{r}_i(r_i + dr_i, \theta_i + d\theta_i), \forall i = 1, 2, 3, 4$ is approximated by Eq. (3), then Eq. (6) can be rewritten as
\[ r_1 e^{j \theta_1} + r_2 d \theta_1 e^{j \theta_1} + dr_1 e^{j \theta_1} + r_3 e^{j \theta_2} + r_4 d \theta_2 e^{j \theta_2} + dr_2 e^{j \theta_2} + r_5 e^{j \theta_3} + r_6 d \theta_3 e^{j \theta_3} \]
\[+ dr_3 e^{j \theta_3} - r_7 e^{j \theta_4} - r_8 d \theta_4 e^{j \theta_4} - dr_4 e^{j \theta_4} = H. \]

Substituting Eq. (4) in Eq. (7), we can obtain in general form that
\[ r_1 d \theta_1 e^{j \theta_1} + dr_1 e^{j \theta_1} + r_2 d \theta_2 e^{j \theta_2} + dr_2 e^{j \theta_2} + r_3 d \theta_3 e^{j \theta_3} + dr_3 e^{j \theta_3} - r_4 d \theta_4 e^{j \theta_4} - dr_4 e^{j \theta_4} = H. \]

Equation (8) represents the residue of vector variations and by means of this equation, \( d \theta_3 \) and \( d \theta_4 \) can be solved.

### 3.4. Proposed Method for Solving \( d \theta_3 \) and \( d \theta_4 \)

To solve Eq. (8) and determine variations \( d \theta_3 \) and \( d \theta_4 \), it is proposed the following hypothesis: the sum of the maximum and minimum residues (residue function \( H \)) with respect to each parameter or variation must be zero. Then we can apply the hypothesis as follows such that
\[
\frac{\partial H}{dr_1} + \frac{\partial H}{dr_2} + \frac{\partial H}{dr_3} + \frac{\partial H}{d \theta_1} + \frac{\partial H}{d \theta_2} + \frac{\partial H}{d \theta_3} + \frac{\partial H}{d \theta_4} = 0. \tag{9}
\]

All partial derivatives resulting from Eq. (9) and which are applied to Eq. (6) can be expressed as follows
\[
[A] \{ \Delta \theta \} + [B] \{ \Delta u \} = 0, \tag{10}
\]
where
\[
[A] = \begin{bmatrix}
-\sin(\theta_1) & -\sin(\theta_2) & \sin(\theta_3) \\
\cos(\theta_1) & \cos(\theta_2) & -\cos(\theta_3)
\end{bmatrix}, \quad
[B] = \begin{bmatrix}
-\sin(\theta_1) + r_1 \cos(\theta_1) & \sin(\theta_2) - r_2 \cos(\theta_2) \\
-r_3 \sin(\theta_3) + \cos(\theta_3) & r_4 \sin(\theta_4) - \cos(\theta_4)
\end{bmatrix}.
\]

\(
\{ \Delta x \} = [d r_1, dr_2, dr_3, d \theta_1, d \theta_2]^T, \quad \text{and} \quad
\Delta u = [d \theta_3, d \theta_4]^T.
\)

With Eq. (10), we can determine the parameters of \( \Delta u \) applying linear algebra such that
\[
\{ \Delta u \} = -[B]^{-1} [A] \{ \Delta x \} = [S_p] \{ \Delta x \}, \tag{11}
\]
where \([S_p]\) is called by Leishman and Chase (2010) as sensibility matrix. Equation (11) is a deterministic model for the variations \( \{ \Delta u \} \), and Wittwer et al. (2004) call this model as the worst case. Additionally, Leishman and Chase (2010) proposed a change in Eq. (11) using a statistic model of the following form
\[
\{ \Delta u \} = \sqrt{\sum (S_p \Delta x_i)^2}. \tag{12}
\]

It is important to mention that Eq. (12) represents a square statistic means of the variations.

### 3.5. Numerical Example for Item 3.1.

For the planar mechanism shown in Fig. 2a (see item 3.1) are assigned numerical values from Table 1 (Example suggested by Leishman and Chase (2010))
Table 1. Dimensions and variations of the mechanism of the Fig. 2a)

<table>
<thead>
<tr>
<th>variable</th>
<th>valor</th>
<th>variation</th>
</tr>
</thead>
<tbody>
<tr>
<td>( r_1 )</td>
<td>5 cm</td>
<td>0.02 cm</td>
</tr>
<tr>
<td>( r_2 )</td>
<td>2 cm</td>
<td>0.01 cm</td>
</tr>
<tr>
<td>( r_3 )</td>
<td>5 cm</td>
<td>0.02 cm</td>
</tr>
<tr>
<td>( r_4 )</td>
<td>4.5 cm</td>
<td>0.015 cm</td>
</tr>
<tr>
<td>( \theta_1 )</td>
<td>( \pi ) rad</td>
<td>0 rad</td>
</tr>
<tr>
<td>( \theta_2 )</td>
<td>0–2( \pi ) rad</td>
<td>0–0.017 rad</td>
</tr>
</tbody>
</table>

4. RESULT AND DISCUSSION

4.1. Results varying \( d\theta_2 \in (0,0.0017) \) rad

Applying the method proposed from Equations (11) and (12), we obtain a field of variations that shows how \( d\theta_3 \) and \( d\theta_4 \) changes in all positions of the mechanism for \( d\theta_2 \in (0,0.0017) \) rad. This is shown in figures 3 and 4.

![Figure 3](image_url)

Figure 3. a) Deterministic model for \( \theta_3 \). b) Deterministic model improved statistically for \( \theta_3 \).
Figure 4. a) Deterministic model for $\theta_j$. b) Deterministic model improved statistically for $\theta_i$.

Figure 3a) shows the values of the variation $\delta\theta_j$ about the nominal value of $\theta_j$, these values was determined with the deterministic solution shown in Eq. (11). Figure 3a) shows that when the angular variation $\delta\theta_j$ takes values bigger than the values of $d\theta_j$ the angular positions augment proportionally. Then, the field of angular variations could define errors in the angular position of $\theta_j$. These variations were determined with Eq.(12). Figure 3b) shows a square means for all values obtained in Fig. 3a) and in this figure, it can be observed that all angular variations are equal to each value of $d\theta_j$. Therefore, we point out that it is possible to establish with this result a tolerance for the angular position $\theta_j$, if we know the range of the angular variation in the input (in this case $d\theta_j$). Similarly, in Fig. 4 the results shown for the field of variations about $d\theta_4$ present a behavior analogous to Fig 3.

**4.2. Results** $d\theta_2 = 0.0017\text{rad}$

A specific case is shown in Fig. 5a) and 5b) such that $d\theta_2 = 0.0017\text{rad}$. In this figure we can observe that angular variations are bigger in the Fig. 5a) in the majority of the range of $\theta_2$. On the other hand, if a statistic mean (see Eq. (12)) is applied to the deterministic solution (see Eq. (11)) the angular variations obtained with the proposed solution are smaller than the ones obtained with DLM.
5. CONCLUSIONS

This study shows a new method for determining position errors in planar mechanisms including dimensional variation in its linkages and orientation of it. The established method in this study shows an agreement with others methods proposed in the literature, for instance DLM. The method was applied to a particular example such as four bar mechanism. The result obtained for the angular position shows that the proposed method determines bigger errors in the angular position than the DLM. Using the deterministic model and if we apply the deterministic model improved statistically the results are opposite. The position errors determined with this method can be used by the designers of precision mechanism for projecting errors in the angular position when the mechanisms are manufactured.

6. ACKNOWLEDGMENTS

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7. REFERENCES